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Determinism in Space-Time

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Generalizing a definition given by Budic and Sachs we define the set $\Theta(M)$ of deterministic points of a space-time M, and show that $\Theta(M) \neq \varnothing$ implies that M admits compact achronal slices. Further we give a new characterization of space-times with $\Theta(M)=M$. The relation between determinism, existence of particle horizons, and visible Cauchy surfaces is investigated.

INTRODUCTION

Most frequently determinism is understood in the sense of global hyperbolicity which guarantees a unique solution of the initial value problem in general relativity (Hawking and Ellis, 1973: HE). In practice one needs the knowledge of data on a Cauchy surface, the existence of which is equivalent to global hyperbolicity (Geroch, 1970). However, for any observer this knowledge is obtainable only if the Cauchy surface is contained in the past light cone of an event on his world line. This leads to the second concept of determinism introduced by Budic and Sachs (BS). They considered space-times with the property that each event can in a certain sense predict its future from its past. It was shown in BS (1976a) that this BS-determinism severely restricts the structure of the past infinity. Particularly BS-deterministic space-times have no particle horizons, a property originally envisaged for Misner's Mixmaster universe (Misner, 1969). Since there is no a priori reason to demand BS-determinism for the whole of space-time, in Section 1 we consider more generally the set $\Theta(M)$ of deterministic points for a space-time M and show that $\Theta(M) \neq \varnothing$ implies that M is closed in the sense of admitting compact achronal slices. Section 2 deals with BS-deterministic space-times and provides proofs for proposi-

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tions stated in BS (1976) without proof. Furthermore, a new characterization of these space-times is given.

In Section 3 we briefly consider the relation between BS-determinism and the existence of particle horizons. Finally, the subject of Section 4 is the connection between the existence of visible Cauchy surfaces in the sense mentioned above and BS-determinism.

Our notation largely follows HE except that we demand a Cauchy surface to be achronal rather than acausal (Geroch, 1970; Penrose, 1972).

A space-time M is understood to be a smooth Hausdorff manifold with a smooth Lorentzian metric g on M such that (M, g) is time orientable.

1. THE SET OF DETERMINISTIC POINTS

We define a space-time M to be *deterministic* in $p \in M$ iff each past-endless nonspacelike curve which intersects $I^+(p)$ also intersects $I^-(p)$. Let $\Theta(M)$ denote the set of deterministic points.

The property of being a deterministic point depends crucially on the global structure of space-time. If an observer reaches a deterministic event of space-time, he will be able to determine his future completely from his past (BS, 1976a). We easily get the following:

Lemma 1.1. $p \in \Theta(M) \Rightarrow J^+(p) \subset \Theta(M)$.

A curve γ contained in the (topological) boundary $\dot{I}^-(p)$ of the past of $p \in M$ is called a *generator* of $\dot{I}^-(p)$ iff γ is a null geodesic such that for any null geodesic $\gamma', \gamma \subset \gamma' \subset \dot{I}^-(p)$ implies $\gamma = \gamma'$ (as subset of M).

Lemma 1.2. For $p \in \Theta(M)$, every generator of $\dot{I}^-(p)$ with future end point p has a past end point in $\dot{I}^-(p)$.

Proof. Let γ be such a generator of $\dot{I}^-(p)$ and γ' denote the maximal continuation of γ into the past. Since $p \in \Theta(M)$, γ' intersects $I^-(p)$. It follows that $\gamma' \cap \dot{I}^-(p)$ is compact. Thus γ has a past end point in $\dot{I}^-(p)$. ∎

It is easy to see that the set of past end points of generators of $\dot{I}^-(p)$ agrees with the null cut locus $C_N^-(p)$ as defined in Beem and Ehrlich (1979) using the Lorentzian distance function. Our next lemma together with Lemma 1.2 implies that for $p \in \Theta(M)$ this set is contained in a compact subset of M.

Lemma 1.3. Let $p \in M$. If every generator of $\dot{I}^-(p)$ has a past endpoint in $\dot{I}^-(p)$, then $E^-(p) = J^-(p) - I^-(p)$ is compact.

Proof. (Compare Lemma 5.1 in BS, 1976b). Let $Y \in T_p(M)$ be a future-pointing timelike vector. We define the following two sets:

$$O(p) := \{X \in T_p(M) | g(X, X) = 0, g(X, Y) = 1\}$$

$$\tilde{O}(p) := \{Z \in T_p(M) | Z = cX, X \in O(p), 0 \leq c \leq f(X)\}$$

where $f(X) \in [0, \infty)$ is defined by $\exp_p[f(X)X] \in C_N^-(p)$. We will show that $\tilde{O}(p)$ is compact. Let $Z_n = c_n X_n$ be a sequence in $\tilde{O}(p)$. Since $O(p)$ is compact there is an accumulation point $X \in O(p)$ of the sequence X_n. Choose a subsequence X_n' which converges to X and denote the corresponding subsequence of c_n by c_n'. Suppose that c_n' has no accumulation point in the interval $[0, f(X)]$. Then there exists a subsequence c_n'' of c_n' with $c_n'' \geq c$ $(\forall n)$ where $c > f(X)$ is fixed and small enough so that $\bar{p} \equiv \exp_p(cX)$ is defined. Since $\bar{p} \in I^-(p)$, there is an integer n_0 such that $n \geq n_0$ implies $\exp_p(c_n'' X_n'') \in I^-(p)$. But this contradicts $c_n'' \leq f(X_n'')$. Hence every sequence in $\tilde{O}(p)$ has an accumulation point in $\tilde{O}(p)$ and $\tilde{O}(p)$ is compact. Since the exponential map $\exp_p$ is continuous, the set $E^-(p) = \exp_p[\tilde{O}(p)]$ is compact too. ■

If $C_N^-(p)$ consists of a single point only, then M admits a spacelike S^{m-1} hypersurface where $m = \dim(M)$ (Rosquist, 1980). It would be interesting to have corresponding results for the general case.

Now we formulate our main result of the section:

> *Theorem 1.4.* Let strong causality hold on M and $\Theta(M) \neq \varnothing$. For each $q \in \operatorname{int}(\Theta(M))$, $E^-(q)$ is a connected component of $\dot{I}^-(q)$ with edge $(E^-(q)) = \varnothing$.

Proof. For $q \in \operatorname{int}(\Theta(M))$ choose a point $p \in \Theta(M) \cap I^-(q)$. Let $r \in E^-(q)$ and λ a past-endless timelike curve which passes through r. Since $q \in I^+(p)$, λ can be chosen so that $\lambda \cap I^+(p)$ is not empty. Then $p \in \Theta(M)$ yields $\lambda \cap \dot{I}^-(p) \neq \varnothing$ and $r \in I^+(\dot{I}^-(p))$. Thus $E^-(q) \subset I^+(\dot{I}^-(p))$. Suppose now there is a point $s \in (\dot{I}^-(q) - E^-(q)) \cap I^+(\dot{I}^-(p))$. Choose a generator λ of $\dot{I}^-(q)$ through the point s. λ could have a future end point only at q (see HE, p. 188), but this would contradict $s \notin E^-(q)$. Thus $J^+(s) \cap J^-(q)$ is not compact. Using Proposition 6.6.2 in HE we can find a nonspacelike curve without past end point which intersects $I^+(p)$ but not $I^-(p)$ in contradiction to $p \in \Theta(M)$.

We see that the compact set $E^-(q)$ is contained in the open set $I^+(\dot{I}^-(p))$ which does not intersect the complement of $E^-(q)$ in $\dot{I}^-(q)$. Since $E^-(q)$ is connected, this shows that $E^-(q)$ is a connected component of $\dot{I}^-(q)$. With edge $(\dot{I}^-(q)) = \varnothing$ (see Penrose, 1972) we get edge $(E^-(q)) = \varnothing$. ■

Hence each space-time satisfying the strong causality condition and containing deterministic points admits compact achronal slices (connected sets without edge) which are past trapped (see HE, p. 267). We would like to mention that a result of Ishikawa could be used to show that $\Theta(M) \neq \varnothing$ for a causally continuous space-time implies a simple causal structure which especially rules out certain types of "trouser worlds" (see Ishikawa, 1977; Hawking and Sachs, 1974).

2. BS-DETERMINISTIC SPACE-TIMES

A space-time M is called BS-*deterministic* iff $\Theta(M) = M$ (BS 1976). The following theorem shows that BS-determinism implies the older notion of determinism mentioned in the introduction.

Theorem 2.1 (BS 1976). Let the strong causality condition hold on M. Then BS-determinism implies global hyperbolicity.

Proof. Suppose M is not globally hyperbolic. From Proposition 6.6.2 in HE it follows that there is a point $p \in M$ and a nonspacelike curve λ in $J^+(p)$ without past end point in M. Since M is deterministic in p, λ has to intersect $I^-(p)$. But then $J^+(p) \cap I^-(p) \neq \varnothing$ and M would violate the causality condition. ■

For a nonspacelike curve λ without past end point the set $\dot{I}^+(\lambda)$ is called the *creation-horizon* of λ. $I^+(\lambda)$ is a TIF iff it cannot be written in the form $I^+(p)$ with $p \in M$ (Geroch et al., 1972). The past infinity of a space-time M can be described by the set $\check{M}$ of TIFs. For a BS-deterministic space-time, the set $\check{M}$ consists of exactly one TIF according to the next theorem due to Budic and Sachs.

Theorem 2.2 (BS 1976a). For a distinguishing space-time M the following conditions are equivalent:

(1) $\Theta(M) = M$;
(2) M has exactly one TIF $(=M)$;
(3) M has no (nonempty) creation horizon.

The Einstein static universe and the Eddington–Lemaitre universe are simple examples of BS-deterministic space-times. This follows from Theorem 2.2 and inspection of the corresponding Penrose diagrams (see, e.g., Penrose, 1978, p. 224). The next theorem states that BS-determinism is much stronger than global hyperbolicity which is equivalent to the existence of a Cauchy surface (Geroch, 1970).

Theorem 2.3 (BS 1976a). Let M be strongly causal and $\Theta(M) = M$. $\dot{I}^-(p)$ is then a compact Cauchy surface $(\forall p \in M)$.

Proof. Suppose there is a timelike curve λ in M without end point such that $\lambda \cap \dot{I}^-(p) = \varnothing$. If $\lambda \subset I^-(p)$ then for each $q \in \lambda$ we get from Proposition 6.4.7 in HE that $J^+(q) \cap J^-(p)$ is not compact. This contradicts Theorem 2.1. If $\lambda \cap \overline{I^-(p)} = \varnothing$ we can choose a timelike curve γ in $I^-(p)$ without past end point. Then $I^+(\lambda)$ and $I^+(\gamma)$ are different TIFs, which contradicts Theorem 2.2. Since $\dot{I}^-(p)$ is achronal it follows that every timelike curve without end point intersects $\dot{I}^-(p)$ exactly once. Thus $\dot{I}^-(p)$ is a compact Cauchy surface by Lemmata 1.2 and 1.3. Note that $E^-(p) = \dot{I}^-(p)$ (Proposition 6.6.1 in HE). ■

Now we will give a further characterization of BS-deterministic space-times:

> *Theorem 2.4.* Let M be strongly causal. $\Theta(M) = M$ iff every past-endless null geodesic γ with future end point fulfills one (or both) of the following conditions:
>
> (1) γ contains a pair of conjugate points (see HE).
>
> (2) There exists another null geodesic γ' joining the future end point of γ to another point on γ.

Proof. "$\Rightarrow$": This follows from Theorem 5.3 in (Beem and Ehrlich, 1979) together with Lemma 1.2 and Theorem 2.1.

"$\Leftarrow$": Suppose M is not BS-deterministic. From Theorem 2.2 we see that there exists a TIF $F \neq M$. Then $\dot{F} \neq \varnothing$ and $\dot{F}$ is generated by past-endless null geodesics (Geroch et al., 1972). Let γ be such a generator. If γ satisfies (1) we can use Proposition 4.5.12 in HE in order to get a contradiction to the achronality of $\dot{F}$. If γ satisfies (2) we can choose a point $q \in \gamma \cap \gamma'$, $q \neq p$. But then p and each point $r \in \gamma$ which lies in the past of q can be joined by a nonspacelike curve which is not a null geodesic. Hence it follows from Proposition 4.5.10 in HE that r lies in $I^-(p)$, in contradiction to the achronality of $\dot{F}$. ■

In order to decide whether a given space-time model is BS-deterministic, we have to consider the set of past-endless null geodesics starting from a fixed but arbitrary chosen point. If for each such null geodesic there is another (possibly infinitesimally neighboring) null geodesic in this set which intersects the former, then the space-time is BS-deterministic. Condition (2) in Theorem 2.4 is just the possibility of "looking around the universe."

3. DETERMINISM AND PARTICLE HORIZONS

In Rindler (1956) a definition of particle horizon for Robertson–Walker universes was introduced. Actually this definition can be applied to every space-time with a cosmic time, i.e., a stably causal space-time (see HE).

From Rindler's definition we immediately obtain a necessary and sufficient condition for the existence of particle horizons, i.e., there are timelike geodesics γ and γ' without past end point, and a point p on γ so that $\gamma' \cap J^{-}(p) = \varnothing$. We see that nonexistence of creation horizons implies nonexistence of particle horizons. Using Theorem 2.2 we find that a BS-deterministic space-time does not admit a particle horizon.

As a necessary and sufficient existence condition for particle horizons in a Robertson–Walker universe, Rindler obtained the convergence of the integral

$$\int_0^{t_0} R(\tau)^{-1} d\tau$$

or

$$\int_{-\infty}^{t_0} R(\tau)^{-1} d\tau$$

respectively, where $R(t)$ is the usual world radius function. Since for every solution $R(t)$ of the Friedmann equation in general relativity (see Rindler, 1977) which vanishes at $t=0$ the integral above converges, we have no BS-deterministic Friedmann universe with a big bang. It is easy to find other closed Friedmann universes with $\Theta(M)=M$. We refer to (BS, 1976a) for further examples.

Let us now consider the metric $ds^2 = -dt^2 + t^2\, d\phi^2$, where $t>0$ and ϕ is taken mod 4π. Null geodesics are given by $(t, \phi_0 \pm \ln(t/t_0))$. Thus past-directed null geodesics starting from an arbitrarily chosen point circumnavigate this two dimensional universe infinitely often. Clearly the two null geodesics starting from a point p will intersect each other in the past. Hence this space-time is BS-deterministic according to our Theorem 2.4. In particular it has no particle horizon. The metric above forms part of the special Kasner solution

$$ds^2 = -dt^2 + t^2\, dx^2 + dy^2 + dz^2$$

where x, y, z are taken mod 4π. Thus we can say there are "no particle horizons in the x direction" in this universe. Misner's Mixmaster universe (Misner, 1969) approximates this model periodically with the axis of horizon removal changing. This suggests that there will be no particle horizons in the Mixmaster universe at all. Consequently this model would explain the observed isotropy of the 3 K-radiation (see Misner, 1969). However, more detailed investigations found it very unlikely that particle horizons will be absent (Doroshkevich and Novikov, 1971; Chitre, 1972).

4. SPACE-TIMES WITH VISIBLE CAUCHY SURFACES

A *visible Cauchy surface* is a Cauchy surface which is contained in $I^-(p)$ for an event $p \in M$.

Lemma 4.1. If $I^-(p)$ contains a Cauchy surface then $\dot{I}^-(p)$ is a Cauchy surface.

Proof. Suppose there is a timelike curve γ without endpoint which does not intersect $\dot{I}^-(p)$. Since γ has to intersect the Cauchy surface contained in $I^-(p)$ we get $\gamma \subset I^-(p)$. By use of Proposition 6.4.7 in HE either M is not strongly causal or $J^+(q) \cap J^-(p)$ is not compact. Hence M is not globally hyperbolic, in contradiction to our assumption. Since $\dot{I}^-(p)$ is achronal and edgeless, it is a Cauchy surface. ■

The next lemma shows the relation between visible Cauchy surfaces and the set of deterministic points.

Lemma 4.2. If $\dot{I}^-(p)$ is a Cauchy surface in M, then $I^+(p) \subset \Theta(M)$.

Proof. Let γ be a nonspacelike curve without past endpoint which intersects $I^+(q)$, $q \in I^+(p)$. Since $\dot{I}^-(p)$ is a Cauchy surface, γ has to intersect $\dot{I}^-(p)$ (see Geroch, 1970, Property 3). $\dot{I}^-(p) = E^-(p)$ yields $\dot{I}^-(p) \subset I^-(q)$. Hence γ intersects $I^-(q)$ and we get $q \in \Theta(M)$. ■

From Lemmata 4.1 and 4.2 and 1.2 and 1.3, we obtain the following:

Theorem 4.3. If $I^-(p)$ contains a Cauchy surface, then $\dot{I}^-(q)$ is a compact Cauchy surface $\forall q \in I^+(p)$.

The following theorem is stated in BS (1976a) without proof:

Theorem 4.4. Let M be strongly causal. The following conditions are equivalent:

(1) $\Theta(M) = M$;
(2) $\forall p \in M: \dot{I}^-(p)$ is a Cauchy surface;
(3) $\forall p \in M: I^-(p)$ contains a Cauchy surface.

Proof.
(1)⇔(2): Theorem 2.3 and Lemma 4.2.
(2)⇒(3): $q \in I^-(p)$ implies that $\dot{I}^-(q)$ is a Cauchy surface contained in $I^-(p)$.
(3)⇒(2): Lemma 4.1. ■

5. CONCLUSION

Although BS-determinism seems too strong an assumption for restricting space-time models for our universe, the possibility remains that our galaxy has already reached or at least will reach in the future a set of deterministic events. We presume that demanding global hyperbolicity (nonexistence of "naked singularities") then provides visible Cauchy surfaces.

In any case we would have to conclude that our universe is closed (see Fliche and Souriau, 1979; Gunn and Tinsley, 1975; for corresponding arguments).

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